QUANTUM GROUPS AND REPRESENTATIONS WITH HIGHEST WEIGHT

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April 8, 1997

Abstract. We consider a special category of Hopf algebras, depending on parameters Σ which possess properties similar to the category of representations of simple Lie group with highest weight λ. We connect quantum groups to minimal objects in this categories—they correspond to irreducible representations in the category of representations with highest weight λ. Moreover, we want to correspond quantum groups only to finite dimensional irreducible representations. This gives us a condition for λ: λ— is dominant means the minimal object in the category of representations with highest weight λ is finite dimensional. We put similar condition for Σ. We call Σ dominant if the minimal object in corresponding category has polynomial growth. Now we propose to define quantum groups starting from dominant parameters Σ.

1. Definitions and examples

1.1 Torus. Let us fix an n-dimensional torus H (i.e. an algebraic group isomorphic to $\mathbb{C}^{*n}$). We denote by S the Hopf algebra of regular functions on H. Let Λ be the lattice of characters of H. Then $\Lambda \subset S$ is a basis in S. The dual algebra S^* can be realized as the algebra of functions on the lattice Λ. We denote by $\hat{H}$ the group algebra of H: for any element $h \in H$ we denote the corresponding generator in $\hat{H}$ by $\hat{h}$ ($\hat{h}_1\hat{h}_2 = \widehat{h_1h_2}$). The Hopf algebra $\hat{H}$ is a subalgebra in S^* ($\hat{h}(\lambda) = \lambda(h)$) on which the comultiplication is well defined: $\Delta\hat{h} = \hat{h} \otimes \hat{h}$.

The Hopf algebra $\hat{H}$ is too small for some of our future purposes. In order to be able to define the comultiplication on S^* we have to complete $S^* \otimes S^*$ to $(S \otimes S)^*$. We need the comultiplication to define an action of S^* on the tensor product of two S^*-modules. We are interested only in those S^*-modules W for which the S^*-module structure is inherited from some S-comodule structure. That means, we should consider only S^*-modules W which are algebraic representations of H. Under this condition the completed comultiplication on S^* would allow us to define the action of S^* on the tensor product of two such S^*-modules (see [B-Kh]).

We set $S^*\hat{\otimes}S^* := (S \otimes S)^*$. The algebra $S^*\hat{\otimes}S^*$ could be realized as the algebra of all functions on the lattice $\Lambda \oplus \Lambda$. If $f \in S^*$ then $\Delta f(\lambda_1, \lambda_2) = f(\lambda_1 + \lambda_2)$.

For convenience, let us denote by $S^\star$ either $\hat{H}$, or S^* with restrictions described above (or, any other suitable representative of a dual Hopf algebra of S)

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1.2 Datum. Given H our datum Σ would be two finite sets of size m: $\{\alpha_1, \alpha_2, ..., \alpha_m\} \subset \Lambda \setminus \{0\}$ — the set of non-zero characters; and $\{\gamma_1, \gamma_2, ..., \gamma_m\} \subset H$ — the set of points of the torus. We denote a generator $\hat{\gamma}_k \in \hat{H}$ corresponding to the point $\gamma_k \in H$ by K_k. We denote $\alpha_i(\gamma_j)$ by q_{ij}.

1.3 Tetramodules. Using our datum we can construct an S-tetramodule T and an $\hat{H}$-tetramodule V (we can consider V as an S^*-tetramodule, see above). For definition of tetramodule (Hopf bimodule etc.) (see [B-Kh] and references there). Informally, S-tetramodule is an S-bimodule and S-bicomodule with some natural axioms. The tetramodule T is generated over S by its space of right H-invariants. The elements $t_i \quad 1 \le i \le m$ would generate a linear basis in this space. Then we describe an S-tetramodule structure of T as follows:

$$\Delta t_i = t_i \otimes 1 + \alpha_i \otimes t_i$$

$$s t_i s^{-1} = s(\gamma_i) t_i \quad \text{for} \quad s \in \Lambda.$$

Analogously, an $\hat{H}$-tetramodule V is generated over $\hat{H}$ by elements $E_i \;\; 1 \le i \le m$ with the following tetramodule structure:

$$\Delta E_i = E_i \otimes 1 + K_i \otimes E_i$$

$$\hat{h} E_i \hat{h}^{-1} = \alpha_i(h) E_i \;\; \text{for} \;\; \hat{h} \in \hat{H} \subset S^*.$$

1.4 Categories. Given an S-tetramodule T denote by $\mathcal{H}(S,T)$ the category of $\mathbb{Z}_+$-graded Hopf algebras B such that $B_0 = S, \; B_1 = T$; and B supplies T with the given S-tetramodule structure.

Given our datum we have two categories—$\mathcal{H}(S,T)$ and $\mathcal{H}(S^\star, V)$. By latter category we mean either $\mathcal{H}(\hat{H}, V)$, or $\mathcal{H}(S^*, V)$ with restrictions discussed above.

1.5 Examples. 1. Let G be a simple Lie group, and H—its Cartan subgroup. Consider datum Σ depending on a parameter q. We put Σ to be the set $\{\alpha_i\}$ of simple roots and the set $\{\gamma_i\}$ such that $\alpha_i(\gamma_j) = q^{<\alpha_i, \alpha_j>}$, where the elements $< \alpha_i, \alpha_j >$ form a Cartan matrix of G. Then the universal enveloping algebra of a Borel subalgebra B_+ of a quantum group G_q is an object in $\mathcal{H}(S^*, V)$. Actually, when $q \ne 1$ the Hopf algebra $U(B_+)$ is an object in $\mathcal{H}(\hat{H}, V)$, but the limit $q \to 1$ should be considered in the bigger algebra.

2. Let G be a reductive algebraic group; H—its Cartan subgroup. Let $A = \mathbb{C}[G]$ be the Hopf algebra of regular functions on G and I the Hopf ideal of functions equal to 0 on H. Then $S = \mathbb{C}[H]$ equals A/I; and the adjoint graded Hopf algebra grA (with respect to I) is the object in $\mathcal{H}(S,T)$, where T is described by datum $\{\alpha_i\}$—the set of non-zero roots and all γ_i equals 1.

3. Let G_q be a quantum deformation of a simple Lie group G. By this we mean a flat family of Hopf algebras A_q which are deformations of $A = \mathbb{C}[G]$. It can be shown that we can flatly deform an ideal I (see Example 2) so that the family of quotient Hopf algebras $H_q = A_q/I_q$ is constant and equals $S = \mathbb{C}[H]$. It is easy to see that the adjoint graded algebra grA_q for generic q is the object in $\mathcal{H}(S,T)$, where T is defined by following datum Σ: the set of characters is the set $\{\alpha_i, -\alpha_i\}$, where α_i are all simple roots, the set of points is $\{\gamma_i, \gamma_i\}$ such that $\alpha_i(\gamma_j)$ are defined by Cartan matrix of G.

Given a quantum group G_q we can construct a Hopf algebra $B_q^1 \in \mathcal{H}(S^\star, V)$ for some datum $\Sigma^1(G_q)$, see example 1. Also, there is another construction (from example 3) of a Hopf algebra $B_q^2 \in \mathcal{H}(S, T)$ for some other datum $\Sigma^2(G_q)$. We suggest that we have some construction (of example 1 or 3, or maybe similar) such that we can describe any quantum group G_q in terms of some Hopf algebra B_q, which is an object in the category $\mathcal{H}(S, T)$ or $\mathcal{H}(\hat{H}, V)$ (resp. $\mathcal{H}(S^*, V)$) for some datum Σ.

1.6 Our goals. Given a quantum group G_q we constructed a $\mathbb{Z}_+$-graded Hopf algebra in some category $\mathcal{H}(S, T)$ for some datum Σ (see 1.5). We would like to answer the following questions:

1) Given datum Σ how we can distinguish an object in the category $\mathcal{H}(S, T)$ which could correspond to a quantum group.

2) What properties should Σ satisfy in order to supply the category $\mathcal{H}(S, T)$ with an object which correspond to some quantum group.

2. The category of modules with highest weight

2.1. Our intuition and constructions are partly based on the idea from our previous paper [B-Kh] of deep parallelism of properties of the category $\mathcal{H}(S, T)$ (resp. $\mathcal{H}(S^*, V)$) and the category of modules with highest weight λ.

Our datum now are the simple Lie group G and the weight λ, $\lambda \in \mathfrak{H}^*$. We consider the category $\mathcal{O}'$ of modules over G from the category $\mathcal{O}$, such that all their weights belong to the set $\{\lambda - \Gamma_+\}$, where Γ_+ is generated over $\mathbb{N}$ by simple roots.

2.2. Consider the functor J: $\mathcal{O}' \to \mathcal{V}$, where we denote the category of vector spaces by $\mathcal{V}$, such that to any module M we correspond the vector space X of vectors of weight λ.

Lemma. *The functor J possess the left adjoint functor F: $\mathcal{V} \to \mathcal{O}'$.*

Consequently, for any $X \in \mathcal{V}$ we can construct a module FX, such that

$$\hom_{\mathcal{O}'}(FX, M) = \hom_{\mathcal{V}}(X, JM). \tag{1}$$

Example. If $X = \mathbb{C}$ then $FX = M_\lambda$—the Verma module with highest weight λ which possess the fixed vector of weight λ. The equality (1) means that for any module M and for any morphism $\mathbb{C} \to JM$ we can construct a unique morphism $M_\lambda \to M$ which is identical on the image of $\mathbb{C}$. Denote by ${}^{\mathbb{C}}\mathcal{M}$ the category of modules from $\mathcal{O}'$ together with the fixed morphism $\mathbb{C} \to JM$. Then we can rephrase our example, saying that the Verma module is the initial object in the category ${}^{\mathbb{C}}\mathcal{M}$ (compare [B-Kh]).

Proof. The example above gives us a proof when $\dim X = 1$. The proof for any X could be easily modified from this example.

Lemma. *The functor J possess the right adjoint functor H: $\mathcal{V} \to \mathcal{O}'$.*

Consequently, for any X in $\mathcal{V}$ we can construct a module FX such that:

$$\hom_{\mathcal{V}}(JM, X) = \hom_{\mathcal{O}'}(M, HX). \tag{2}$$

Example. If X=$\mathbb{C}$ then $FX = \delta_\lambda$—the contragredient Verma module with fixed highest covector of weight λ. The equality above means that for any module M and any morphism $JM \to \mathbb{C}$ we can construct a unique morphism $M \to \delta_\lambda$ which is identical on the fixed covector. Denote by $\mathcal{M}^{\mathbb{C}}$ the category of modules from $\mathcal{O}'$ together with the fixed morphism $JM \to \mathbb{C}$. Then we can rephrase our example, saying that the contragredient Verma module δ_λ is the final object in the category $\mathcal{M}^{\mathbb{C}}$ (compare [B-Kh]).

Proof. In the category $\mathcal{O}'$ there is a natural duality: $M \to M^\star$, where we define each weight subspace of $M^\star$ as a regular dual to corresponding weight subspace in M: $(M^\star)_\eta = (M_\eta)^*$. The action of the group G is defined naturally. After this remark the existance of final object in $\mathcal{M}^{\mathbb{C}}$ becomes automatic and the construction of the contragredient Verma module becomes trivial: $\delta_\lambda = (M_\lambda)^\star$.

2.3 Shapovalov map. Consider the category $\mathcal{M}$ of modules $M \in \mathcal{O}'$ such that $\dim(JM) = 1$ together with fixed isomorphism between JM and $\mathbb{C}$. This category is the subcategory in ${}^{\mathbb{C}}\mathcal{M}$ and in $\mathcal{M}^{\mathbb{C}}$. As $M_\lambda \in \mathcal{M} \subset {}^{\mathbb{C}}\mathcal{M}$ it is an initial object in $\mathcal{M}$, analogously, δ_λ is a final object in $\mathcal{M}$. Therefore, there exists a canonical map Sh: $M_\lambda \to \delta_\lambda$ which is called a Shapovalov map. (As $\delta_\lambda \subset M_\lambda^*$ the Shapovalov map defines the Shapovalov form on M_λ).

Going back to functors, if we put $M = FX$ into (1), we get

$$\hom_{\mathcal{O}'}(FX, FX) = \hom_{\mathcal{V}}(X, JFX).$$

The identity isomorphism on the left corresponds to a canonical element j on the right which is called the adjunction map: j: $X \to JFX$. It is easy to check that in the category $\mathcal{M}$ the adjunction map j: $X \to JFX$ is an isomorphism $\forall X \in \mathcal{V}$. If we put instead of M the module FX in (2) we would get:

$$\hom_{\mathcal{O}'}(FX, HX) = \hom_{\mathcal{V}}(JFX, X).$$

For the category $\mathcal{M}$ the identity element in the right hand side (which is inverse of the adjunction map j) would correspond to a canonical morfisms of functors on the left:

Lemma. *In the category $\mathcal{M}$ there exists a canonical morphism of functors $F \to H$.*

We denote L_λ the image of the Shapovalov map in δ_λ. The module L_λ is an irreducible representation of G with the highest weight λ. The module L_λ is the minimal object in $\mathcal{M}$. That means that for any module $B \in \mathcal{M}$ there exists a submodule $B' \subset B$ and a canonical epimorphism $B' \to L_\lambda$.

2.4 Point. We would keep in mind for the next section that minimal object in the category $\mathcal{M}$ is of importance. Also, if we consider the category $\mathcal{M}$ as a function of λ we may say that we are interested in those λ's when the dimension of minimal object in $\mathcal{M}$ is dropping significantly. (Weight λ is called dominant if L_λ is finite dimensional).

3. Our categories

3.1 Parallel construction. Let us fix S—the Hopf algebra of functions over torus. We denote by $\mathcal{T}$ the category of tetramodules over S. We denote by $\mathcal{A}$ the category of $\mathbb{Z}_+$-graded Hopf algebras with 0-graded subalgebra isomorphic to S and the subspace of each grade is finitely generated over S. Consider the functor $J\colon \mathcal{A} \to \mathcal{T}$ which corresponds to a given Hopf algebra A a 1-graded subspace of A with inherited S-tetramodule structure [B-Kh].

Lemma. *The functor J possess the left adjoint functor $F\colon \mathcal{T} \to \mathcal{A}$.*

Proof. Given an S-tetramodule T we can construct a $\mathbb{Z}_+$-graded algebra A such that $A_0 = S,\ A_1 = T$, and the algebra A is freely generated as an algebra by S and T and A supplies T with the given S-bimodule structure. We can construct a comultiplication on A by multiplicativity. Being universal as an algebra, the Hopf algebra A remains universal as a Hopf algebra.

Lemma. *The functor J possess the right adjoint functor $F\colon \mathcal{T} \to \mathcal{A}$.*

Proof. Given an S-tetramodule T we can construct a $\mathbb{Z}_+$-graded coalgebra A such that $A_0 = S,\ A_1 = T$, and the coalgebra A is freely generated as a coalgebra by S and T and A supplies T with the given Sbicomodule structure. We can construct a multiplication on A by comultiplicativity. Being universal as a coalgebra, the Hopf algebra A remains universal, when the Hopf algebra structure is added [B-Kh].

Remark. We can construct a category $\mathcal{A}^\star$ of $\mathbb{Z}_+$-graded Hopf algebras dual to the category $\mathcal{A}$. To each Hopf algebra $A \in \mathcal{A}$ we correspond a Hopf algebra $A^\star \in \mathcal{A}^\star$ such that $A^\star$ as an algebra is a subalgebra in A^* and each component $(A^\star)_n$ of $A^\star$ is isomorphic as a vector space to $S^\star \otimes M^*$, when $A_n = S \otimes M$. After that our universal coalgebra in $\mathcal{A}$ corresponds to universal algebra in $\mathcal{A}^\star$.

As in section 2, let us consider the category $\mathcal{H}(S,T)$ and for each element $A \in \mathcal{H}$ let us fix an isomorphism of A/A_2 with $S \oplus T$. It is easy to see that the category $\mathcal{H}(S,T)$ is similar to the category $\mathcal{M}$ in chapter 2.

In particular, there exists an initial object in the category $\mathcal{H}(S,T)$. It is called a universal algebra. We denote it by $B^i(S,T)$. It corresponds to Verma module M_λ.

There exists a final object in the category $\mathcal{H}(S,T)$. It is called a universal coalgebra. We denote it by $B^f(S,T)$. It corresponds to contragredient module δ_λ.

Hence, there is a canonical map $Sh : B^i \to B^f$ which is analogue of the Shapovalov map. We would denote the image of this map by $B^m(S,T)$. This would be the minimal object in the category $\mathcal{H}(S,T)$. The Hopf algebra B^m corresponds to an irreducible representation in our parallelism.

3.2 Answers to questions. Constructing a parallelism between the categories $\mathcal{M}$ and $\mathcal{H}(S,T)$ we now can use our intuition in $\mathcal{M}$ to understand what is important in $\mathcal{H}(S,T)$. Now we are ready to answer to our first question. If quantum groups corresponds to datum Σ, then it corresponds to a minimal object B^m in the category $\mathcal{H}(S,T)$, where the category $\mathcal{H}(S,T)$ is constructed by our datum.

We would call our datum Σ dominant if the minimal object B^m in the category $\mathcal{H}(S,T)$ has polynomial growth over S.

As an answer to our second question, we suggest that quantum groups correspond to Hopf algebras which are minimal objects constructed from dominant datum.

References

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